

A SUPERSPACE FORMULATION OF THE BV ACTION

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Abstract

We show that the BV (Batalin Vilkovisky) action, formulated with an extended BRST symmetry (including the shift symmetry), is also invariant under an extended anti-BRST transformation (where the antifields are the parameters of the transformation), when the gauge fixing Lagrangian is both BRST and anti-BRST invariant. We show that for a general gauge fixing Lagrangian, the BV action can be written in a manifestly extended BRST invariant manner in a superspace with one Grassmann coordinate whereas it can be expressed in a manifestly extended BRST and anti-BRST invariant manner in a superspace with two Grassmann coordinates when the gauge fixing Lagrangian is invariant under both BRST and anti-BRST transformations.

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I. Introduction

The Batalin-Vilkovisky (BV) formalism [1,2] has proved to be a powerful method of quantization for various gauge theories, as well as supergravity theories and topological field theories in the Lagrangian framework. It encompasses the Faddeev Popov quantization and uses the BRST symmetry, discovered in the context of gauge theories [3], to build on it. It is known that, for standard gauge fixing, gauge theories possess an anti-BRST invariance [4] in addition to the usual BRST symmetry. In contrast, however, the standard discussion of the BV formalism has so far been predominantly in the BRST context [5]. Furthermore, while the BRST and the anti-BRST symmetry of gauge theories can be given a geometrical meaning and have led to a superspace formulation of such theories [6,7], a superspace description of the BV action does not exist so far. The main difficulty behind this lies in the fact that the antifields, introduced in the BV formalism, are identified with the functional derivative of a gauge fixing fermion with respect to the associated field. Thus, a priori, it would appear that a superspace description cannot even be contemplated before choosing a gauge.

Recently, however, it has been shown [8] how an extended BRST invariant formulation (including the shift symmetry) of the BV action, naturally leads to the proper identification of the antifields through equations of motion of auxiliary field variables. This, therefore raises the possibility of a superspace formulation of the BV action which we discuss in this paper. In sec. II we review the extended BRST invariant formulation of the BV action. In sec. III we show, when the gauge fixing Lagrangian is both BRST and anti-BRST invariant, that this action possesses an extended anti-BRST invariance, where the antifields define the transformations. In sec. IV we show how, for a general gauge fixing, the BV action can be written in superspace with a manifest extended BRST invariance. In sec. V we use both the extended BRST and anti-BRST invariances to formulate the BV action in superspace with two Grassmann coordinates and present our conclusions in sec. VI. For clarity of ideas, our entire discussion will be in the context of a non Abelian Yang Mills theory with covariant gauge fixing. The generalization to other systems can be carried out in a straight forward manner.

II. Extended BRST Invariant Formulation

Let us consider a non Abelian Yang Mills theory described by:

$$\mathcal{L} = \mathcal{L}(A_\mu, c, \bar{c}, F) = \mathcal{L}_o(A_\mu) + \mathcal{L}_{gf}(A_\mu, c, \bar{c}, F) \quad (2.1)$$

where $\mathcal{L}_o(A_\mu)$ describes the gauge invariant classical Lagrangian density, while \mathcal{L}_{gf} represents the Lagrangian density associated with gauge fixing and ghosts. We assume that all the fields – the gauge field, ghost, antighost, and the auxiliary field – are matrices belonging to the adjoint representation of the gauge group which we can choose to be $SU(n)$. Thus, for example, for a standard covariant gauge fixing, we have

$$\begin{aligned} \mathcal{L}_o(A_\mu) &= -\frac{1}{4} \text{Tr } F_{\mu\nu} F^{\mu\nu} \\ \mathcal{L}_{gf} &= \text{Tr } (\partial_\mu F A^\mu + \partial_\mu \bar{c} D^\mu c) \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} D_\mu c &= \partial_\mu c + [A_\mu, c] \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \end{aligned} \quad (2.3)$$

and Tr stands for the trace over the $SU(n)$ indices.

As we know, the Lagrangian density in (2.1) is invariant under the BRST transformations

$$\begin{aligned} \delta A_\mu &= \omega D_\mu c \\ \delta c &= -\frac{\omega}{2} [c, c]_+ \\ \delta \bar{c} &= -\omega F \\ \delta F &= 0 \end{aligned} \quad (2.4)$$

where ω is a constant Grassmann parameter. In the BV formalism \mathcal{L}_{gf} is represented as a BRST variation of a general fermion which results in the BRST invariance of the theory. It can be easily checked that the choice of gauge fixing in (2.2) can also be represented as (without the BRST parameter)

$$\mathcal{L}_{gf} = \delta\psi \quad (2.5)$$

where the gauge fixing fermion has the form

$$\psi = -\text{Tr } \partial_\mu \bar{c} A^\mu \quad (2.6)$$

It is straight forward to check that the Lagrangian in Eq. (2.1) is also invariant under the anti-BRST transformations [4]

$$\begin{aligned} \bar{\delta} A_\mu &= \bar{\omega} D_\mu \bar{c} \\ \bar{\delta} c &= \bar{\omega} (F - [c, \bar{c}]_+) \\ \bar{\delta} \bar{c} &= -\frac{\bar{\omega}}{2} [\bar{c}, \bar{c}]_+ \\ \bar{\delta} F &= \bar{\omega} [F, \bar{c}] \end{aligned} \quad (2.7)$$

where $\bar{\omega}$ is the constant Grassmann parameter of the transformation. We note that the gauge fixing fermion in Eq. (2.6) can be written as (without the parameter)

$$\psi = -\frac{1}{2} \bar{\delta} \text{Tr } A_\mu A^\mu \quad (2.8)$$

This, of course, implies that the Lagrangian is both BRST and anti-BRST invariant since the transformations in Eqs. (2.4) and (2.7) are nilpotent. However, we also note that not all gauge fixing fermions which lead to BRST and anti-BRST invariance can be written as the anti-BRST variation of a boson. Furthermore, not all gauge fixing Lagrangians will be invariant under both BRST and anti-BRST transformations [9].

The extended BRST invariant formulation of the BV action is obtained by considering the Lagrangian density [8]

$$\hat{\mathcal{L}} = \mathcal{L}(A_\mu - \tilde{A}_\mu, c - \tilde{c}, \bar{c} - \tilde{\bar{c}}, F - \tilde{F}) \quad (2.9)$$

which coincides with Eq. (2.1) when all the tilde fields vanish. This Lagrangian density is, of course, invariant under the BRST transformation of (2.3) with respect to the fields $\phi - \tilde{\phi}$ where $\phi = \{A_\mu, c, \bar{c}, F\}$. But in addition, it is also invariant under the local shift symmetry

$$\begin{aligned} \delta \phi(x) &= \alpha(x) \\ \delta \tilde{\phi}(x) &= \alpha(x) \end{aligned} \quad (2.10)$$

which needs to be gauge fixed and, in turn, leads to an additional BRST symmetry. Together, the BRST symmetries are commonly referred to as an extended BRST symmetry with the transformations:

$$\begin{aligned}
\delta A_\mu &= \omega \psi_\mu \quad , \quad \delta \tilde{A}_\mu = \omega (\psi_\mu - D_\mu^{(A-\tilde{A})} (c - \tilde{c})) \\
\delta c &= \omega \epsilon \quad , \quad \delta \tilde{c} = \omega \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \\
\delta \bar{c} &= \omega \bar{\epsilon} \quad , \quad \delta \tilde{\bar{c}} = \omega (\bar{\epsilon} + (F - \tilde{F})) \\
\delta F &= \omega \epsilon_F \quad , \quad \delta \tilde{F} = \omega \epsilon_F \\
\delta \psi_\mu &= 0 \\
\delta \epsilon &= 0 \\
\delta \bar{\epsilon} &= 0 \\
\delta \epsilon_F &= 0 \\
\delta A_\mu^* &= -\omega B_\mu \\
\delta B_\mu &= 0 \\
\delta c^* &= -\omega B \\
\delta B &= 0 \\
\delta \bar{c}^* &= -\omega \bar{B} \\
\delta \bar{B} &= 0 \\
\delta F^* &= -\omega B_F \\
\delta B_F &= 0
\end{aligned} \tag{2.11}$$

Here ψ_μ , ϵ , $\bar{\epsilon}$ and ϵ_F are the ghost fields associated with the shift symmetries for A_μ , c , \bar{c} and F respectively, A_μ^* , c^* , \bar{c}^* and F^* are the respective antighosts and B_μ , B , \bar{B} and B_F represent the corresponding auxiliary fields. We note that there is a certain amount of arbitrariness in defining the transformations in Eq. (2.11), but this is the conventional choice.

If we gauge fix the shift symmetry such that all the tilde fields vanish, then, of course, we will recover our original theory. This can be achieved by choosing a gauge fixing Lagrangian of the form:

$$\begin{aligned}
\tilde{\mathcal{L}}_{gf} = & \text{Tr} \left[-B_\mu \tilde{A}^\mu - A_\mu^* (\psi^\mu - D^{\mu(A-\tilde{A})}(c - \tilde{c})) \right. \\
& - \overline{B} \tilde{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \\
& \left. + B \tilde{c} - c^* (\bar{\epsilon} + (F - \tilde{F})) + B_F \tilde{F} + F^* \epsilon_F \right]
\end{aligned} \tag{2.12}$$

It is clear that integrating out the auxiliary fields B_μ, B, \overline{B} and B_F would set all the tilde fields equal to zero and it is also straight forward to check that the gauge fixing Lagrangian, in Eq. (2.12), for the shift symmetry is invariant under the extended BRST transformations of Eq. (2.11). In addition, of course, we have the gauge fixing Lagrangian for the original gauge symmetry (see Eqs. (2.5) and (2.6)). Being the extended variation of a fermion, this is also invariant under the transformations in (2.11) which are nilpotent. If we choose the gauge fixing fermion to depend on the original fields only, then a general gauge fixing Lagrangian for the original symmetry will have the form (we use left derivatives.)

$$\begin{aligned}
\mathcal{L} = \delta\psi = & \text{Tr} \left(\delta A_\mu \frac{\delta\psi}{\delta A_\mu} + \delta c \frac{\delta\psi}{\delta c} + \delta \bar{c} \frac{\delta\psi}{\delta \bar{c}} + \delta F \frac{\delta\psi}{\delta F} \right) \\
= & \text{Tr} \left(-\frac{\delta\psi}{\delta A_\mu} \psi^\mu + \frac{\delta\psi}{\delta c} \epsilon + \frac{\delta\psi}{\delta \bar{c}} \bar{\epsilon} - \frac{\delta\psi}{\delta F} \epsilon_F \right)
\end{aligned} \tag{2.13}$$

Thus, after integrating out the auxiliary fields which set the tilde fields to zero, we have

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_o(A_\mu) + \tilde{\mathcal{L}}_{gf} + \mathcal{L}_{gf} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} \\
& + \text{Tr} \left(A_\mu^* D^\mu c + \frac{1}{2} \bar{c}^* [c, c]_+ - c^* F - \left(A_\mu^* + \frac{\delta\psi}{\delta A_\mu} \right) \psi^\mu \right. \\
& \left. + \left(\bar{c}^* + \frac{\delta\psi}{\delta c} \right) \epsilon - \left(c^* - \frac{\delta\psi}{\delta \bar{c}} \right) \bar{\epsilon} + \left(F^* - \frac{\delta\psi}{\delta F} \right) \epsilon_F \right)
\end{aligned} \tag{2.14}$$

If we now integrate out the ghosts associated with the shift symmetry, it leads to the identification

$$\begin{aligned}
A_\mu^* &= -\frac{\delta\psi}{\delta A_\mu} \\
\bar{c}^* &= -\frac{\delta\psi}{\delta c} \\
c^* &= \frac{\delta\psi}{\delta \bar{c}} \\
F^* &= \frac{\delta\psi}{\delta F}
\end{aligned}
\tag{2.15}$$

For the covariant gauge fixing of Eq. (2.6), this yields

$$\begin{aligned}
A_\mu^* &= \partial_\mu \bar{c} \\
\bar{c}^* &= 0 \\
c^* &= \partial_\mu A^\mu \\
F^* &= 0
\end{aligned}
\tag{2.16}$$

With the identification of Eq. (2.15) (or (2.16) in the particular case), we recover the BV action for the theory [1,2].

III. Extended Anti BRST Invariance

Let us next consider a gauge fixing Lagrangian (such as in Eq. (2.2)) which is invariant under both the BRST and anti-BRST transformations of Eqs. (2.4) and (2.7). In this case, the complete Lagrangian is also invariant under BRST and anti-BRST transformations. The parameter of transformation for the anti-BRST transformations (see Eq. (2.7)) is the antighost field. While the anti-BRST invariance does not lead to any additional information beyond what BRST invariance provides, it helps put the theory in a proper geometrical setting. It is, therefore, interesting to ask whether such a Lagrangian with an extended BRST invariance is also symmetric under an extended anti-BRST transformation. In fact, it is stright forward to check that the transformations

$$\begin{aligned}
\bar{\delta} A_\mu &= \bar{\omega}(A_\mu^* + D_\mu^{(A-\tilde{A})}(c - \tilde{c})) \quad , \quad \bar{\delta} \tilde{A}_\mu = \bar{\omega} A_\mu^* \\
\bar{\delta} c &= \bar{\omega}(c^* + (F - \tilde{F}) - [c - \tilde{c}, \bar{c} - \tilde{c}]_+) \quad , \quad \bar{\delta} \tilde{c} = \bar{\omega} c^* \\
\bar{\delta} \bar{c} &= \bar{\omega}(\bar{c}^* - \frac{1}{2} [\bar{c} - \tilde{c}, \bar{c} - \tilde{c}]_+) \quad , \quad \bar{\delta} \tilde{\bar{c}} = \bar{\omega} \bar{c}^* \\
\bar{\delta} F &= \bar{\omega}(F^* + [F - \tilde{F}, \bar{c} - \tilde{c}]_+) \quad , \quad \bar{\delta} \tilde{F} = \bar{\omega} F^* \\
\bar{\delta} \psi_\mu &= \bar{\omega} (B_\mu + D_\mu^{(A-\tilde{A})}(F - \tilde{F}) - [D_\mu^{(A-\tilde{A})}(c - \tilde{c}), \bar{c} - \tilde{c}]_+) \\
\bar{\delta} \epsilon &= \bar{\omega}(B - [F - \tilde{F}, c - \tilde{c}] + [\bar{c} - \tilde{c}, (c - \tilde{c})^2]) \\
\bar{\delta} \bar{\epsilon} &= \bar{\omega}(\bar{B} - [F - \tilde{F}, \bar{c} - \tilde{c}]) \\
\bar{\delta} \epsilon_F &= \bar{\omega} B_F \\
\bar{\delta} A_\mu^* &= 0 \\
\bar{\delta} B_\mu &= 0 \\
\bar{\delta} c^* &= 0 \\
\bar{\delta} B &= 0 \\
\bar{\delta} \bar{c}^* &= 0 \\
\bar{\delta} \bar{B} &= 0 \\
\bar{\delta} F^* &= 0 \\
\bar{\delta} B_F &= 0
\end{aligned} \tag{3.1}$$

define a symmetry of the complete Lagrangian. The classical part of the Lagrangian density $\mathcal{L}_o(A_\mu - \tilde{A}_\mu)$, is, of course, invariant under these transformations simply because they represent a gauge transformation in these variables. It is straight forward to check that $\tilde{\mathcal{L}}_{gf}$ is also invariant under these transformations. For a gauge fixing Lagrangian of the original symmetry which is both BRST and anti-BRST invariant, it follows that this must be invariant under the extended anti-BRST transformation at least on-shell where the transformations reduce to anti-BRST transformations. For the covariant gauge fixing choice of Eq. (2.6), invariance on shell is obvious since the gauge fixing Lagrangian is the anti-BRST variation of an operator (see Eq. (2.8)). We will discuss this in more detail in sec. V.

We note here that, as in the case of extended BRST transformations, there is an

arbitrariness in the transformations in Eq. (3.1). However, the present choice of transformations leads to a simple superspace formulation of the theory as we will see in sec. V.

IV. Extended BRST Invariant Superspace Formulation

Let us consider a superspace labelled by the coordinates (x^μ, θ) [6]. In this space, a super-connection 1-form will have the form [7]

$$\omega = \phi_\mu(x, \theta)dx^\mu + \eta(x, \theta)d\theta \quad (4.1)$$

where we assume the component superfields to have the form

$$\begin{aligned} \phi_\mu(x, \theta) &= A_\mu(x) + \theta R_\mu(x) \\ \eta(x, \theta) &= c(x) + \theta R(x) \end{aligned} \quad (4.2)$$

$A_\mu(x)$ and $c(x)$ are assumed to be the gauge fields and ghosts associated with a Yang Mills theory and all the components of the superfields in Eq. (4.2) are assumed to belong to the adjoint representation of the gauge group $SU(n)$. The curvature (field strength) associated with the connection in Eq. (4.1) is given by

$$F = d\omega + \frac{1}{2} [\omega, \omega] \quad (4.3)$$

It is known [6,7] that if we require the components of the field strength to vanish along the θ direction, then it determines the superfields to have the form

$$\begin{aligned} \phi_\mu(x, \theta) &= A_\mu(x) + \theta D_\mu c \\ \eta(x, \theta) &= c(x) - \frac{1}{2} \theta [c, c]_+ \end{aligned} \quad (4.4)$$

In other words, the BRST transformations of $A_\mu(x)$ and $c(x)$, in this case result as a consequence of translations of the coordinate θ . In this formalism, the antighosts and other matter fields have to be introduced as additional superfields of the form (for the antighosts)

$$\bar{\eta}(x, \theta) = \bar{c}(x) - \theta F(x) \quad (4.5)$$

Let us next consider the theory defined by

$$\tilde{\mathcal{L}} = \mathcal{L}(\phi_\mu - \tilde{\phi}_\mu, \eta - \tilde{\eta}, \bar{\eta} - \tilde{\bar{\eta}}) \quad (4.6)$$

and note that when the tilde superfields vanish, this Lagrangian reduces to our original theory. In this case, we can define

$$\begin{aligned} \Phi_\mu(x, \theta) &= \phi_\mu(x, \theta) - \tilde{\phi}_\mu(x, \theta) \\ \Lambda(x, \theta) &= \eta(x, \theta) - \tilde{\eta}(x, \theta) \end{aligned} \quad (4.7)$$

and note that if the field strength associated with the 1-form

$$\Omega = \Phi_\mu dx^\mu + \Lambda d\theta \quad (4.8)$$

vanishes along the θ direction, then we can determine

$$\begin{aligned} \Phi_\mu(x, \theta) &= (A_\mu - \tilde{A}_\mu) + \theta D_\mu^{(A-\tilde{A})}(c - \tilde{c}) \\ \Lambda(x, \theta) &= (c - \tilde{c}) - \frac{1}{2} \theta [c - \tilde{c}, c - \tilde{c}]_+ \end{aligned} \quad (4.9)$$

This, however, does not determine the individual superfields ϕ_μ , $\tilde{\phi}_\mu$, η and $\tilde{\eta}$ uniquely and this is the arbitrariness in the extended BRST transformations that we discussed earlier.

Consistent with the discussion in sec. II, let us choose

$$\begin{aligned} \phi_\mu(x, \theta) &= A_\mu + \theta \psi_\mu \\ \tilde{\phi}_\mu(x, \theta) &= \tilde{A}_\mu + \theta (\psi_\mu - D_\mu^{(A-\tilde{A})}(c - \tilde{c})) \\ \eta(x, \theta) &= c + \theta \epsilon \\ \tilde{\eta}(x, \theta) &= \tilde{c} + \theta \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \\ \bar{\eta}(x, \theta) &= \bar{c} + \theta \bar{\epsilon} \\ \tilde{\bar{\eta}}(x, \theta) &= \tilde{\bar{c}} + \theta (\bar{\epsilon} + (F - \tilde{F})) \end{aligned} \quad (4.10)$$

In addition, let us introduce the superfields

$$\begin{aligned}
\tilde{\phi}_\mu^*(x, \theta) &= A_\mu^* - \theta B_\mu \\
\tilde{\eta}^*(x, \theta) &= c^* - \theta B \\
\tilde{\bar{\eta}}^*(x, \theta) &= \bar{c}^* - \theta \bar{B} \\
\tilde{f}(x, \theta) &= \tilde{F} + \theta \epsilon_F \\
\tilde{f}^*(x, \theta) &= F^* - \theta B_F
\end{aligned} \tag{4.11}$$

It is clear now that with these choices of the superfields, the extended BRST transformations of Eq. (2.11) arise as translations of the θ coordinate.

Let us next note the following relations

$$\begin{aligned}
\frac{\partial}{\partial \theta} \text{Tr } \tilde{\phi}_\mu^* \tilde{\phi}^\mu &= \text{Tr} \left(-B_\mu \tilde{A}^\mu - A_\mu^* (\psi^\mu - D^{\mu(A-\tilde{A})} (c - \tilde{c})) \right) \\
\frac{\partial}{\partial \theta} \text{Tr } \tilde{\eta}^* \tilde{\eta} &= \text{Tr} \left(-\bar{B} \tilde{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \right) \\
-\frac{\partial}{\partial \theta} \text{Tr } \tilde{\eta} \tilde{\eta}^* &= \text{Tr} \left(B \tilde{c} - c^* (\bar{\epsilon} + (F - \tilde{F})) \right) \\
-\frac{\partial}{\partial \theta} \text{Tr } \tilde{f}^* \tilde{f} &= \text{Tr} \left(B_F \tilde{F} + F^* \epsilon_F \right)
\end{aligned} \tag{4.12}$$

Thus, we see that the gauge fixing Lagrangian of Eq. (2.12) for the shift symmetry can be written in this superspace as

$$\tilde{\mathcal{L}}_{gf} = \frac{\partial}{\partial \theta} \text{Tr} \left(\tilde{\phi}_\mu^* \tilde{\phi}^\mu + \tilde{\eta}^* \tilde{\eta} - \tilde{\eta} \tilde{\eta}^* - \tilde{f}^* \tilde{f} \right) \tag{4.13}$$

Being the θ component of a superfield, this is manifestly invariant under the extended BRST transformations of Eq. (2.11).

The gauge fixing Lagrangian for the original symmetry can also be written in this space in a straight forward manner. Let $\psi = \psi(A_\mu, c, \bar{c}, F)$ denote an arbitrary gauge fixing fermion. We assume this to depend only on the original fields. Then, we can define a fermionic superfield as

$$\Psi = \psi + \theta \delta \psi = \psi + \theta \left(-\frac{\delta \psi}{\delta A_\mu} \psi_\mu + \frac{\delta \psi}{\delta c} \epsilon + \frac{\delta \psi}{\delta \bar{c}} \bar{\epsilon} - \frac{\delta \psi}{\delta F} \epsilon_F \right) \tag{4.14}$$

For the gauge choice in Eqs. (2.5) and (2.6), then, this superfield will be of the form

$$\Psi = -\text{Tr } \partial_\mu \bar{c} A^\mu + \theta \text{Tr } (\partial_\mu \bar{c} \psi^\mu + \partial_\mu A^\mu \bar{c}) \quad (4.15)$$

We see that the gauge fixing Lagrangian for the original symmetry can be written as

$$\mathcal{L}_{gf} = \frac{\partial \Psi}{\partial \theta} \quad (4.16)$$

for any arbitrary fermionic gauge fixing term. Once again, being the θ component of a superfield, this is manifestly invariant under the extended BRST transformations of Eq. (2.11).

The complete Lagrangian can now be written as (see ref. [7] for a discussion on the structure of the classical Lagrangian \mathcal{L}_o)

$$\begin{aligned} \hat{\mathcal{L}} &= \mathcal{L}_o(\phi_\mu - \tilde{\phi}_\mu) + \tilde{\mathcal{L}}_{gf} + \mathcal{L}_{gf} \\ &= \mathcal{L}_o(A_\mu - \tilde{A}_\mu) + \frac{\partial}{\partial \theta} \text{Tr} \left(\tilde{\phi}_\mu^* \tilde{\phi}^\mu + \tilde{\eta}^* \tilde{\eta} - \tilde{\eta} \tilde{\eta}^* - \tilde{f}^* \tilde{f} \right) + \frac{\partial \Psi}{\partial \theta} \end{aligned} \quad (4.17)$$

which is manifestly invariant under the extended BRST symmetry and upon elimination of the auxiliary fields and ghosts associated with the shift symmetry, leads to the BV action.

V. Extended BRST and anti-BRST Invariant Superspace Formulation

Let us recall briefly some facts about the superspace formulation of a gauge theory with BRST and anti-BRST invariance [7]. If we define a superspace with coordinates $(x^\mu, \theta, \bar{\theta})$, then in such a space, we can define a super connection 1-form of the kind

$$\omega = \phi_\mu(x, \theta, \bar{\theta}) dx^\mu + \eta(x, \theta, \bar{\theta}) d\theta + \bar{\eta}(x, \theta, \bar{\theta}) d\bar{\theta} \quad (5.1)$$

where ϕ_μ , η and $\bar{\eta}$ are matrices belonging to the adjoint representation of the gauge group $\text{SU}(n)$. The field strength, in this case, is given by

$$F = d\omega + \frac{1}{2} [\omega, \omega] \quad (5.2)$$

and will have components not only along the μ, ν directions, but also along all possible $\theta, \bar{\theta}$ directions. Requiring the field strength to vanish along all extra directions determines the superfields uniquely to be

$$\begin{aligned}\phi_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta D_\mu c + \bar{\theta} D_\mu \bar{c} + \theta \bar{\theta} (D_\mu F - [D_\mu c, \bar{c}]_+) \\ \eta(x, \theta, \bar{\theta}) &= c(x) - \frac{1}{2} \theta [c, c]_+ + \bar{\theta} (F - [c, \bar{c}]_+) + \theta \bar{\theta} (-[F, c] + [\bar{c}, c^2]) \\ \bar{\eta}(x, \theta, \bar{\theta}) &= \bar{c}(x) - \theta F - \frac{1}{2} \bar{\theta} [\bar{c}, \bar{c}]_+ - \theta \bar{\theta} [F, \bar{c}]\end{aligned}\tag{5.3}$$

It is now straight forward to compare with Eqs. (2.4) and (2.7) and note that the BRST and anti-BRST transformations merely correspond, in this formulation, to translations in the θ and $\bar{\theta}$ coordinates respectively. The complete Lagrangian of a gauge theory, with BRST and anti-BRST invariant gauge fixing, can be written in this superspace where the invariances are manifest.

Let us next consider a Lagrangian density

$$\tilde{\mathcal{L}} = \mathcal{L}(\phi_\mu - \tilde{\phi}_\mu, \eta - \tilde{\eta}, \bar{\eta} - \tilde{\bar{\eta}})\tag{5.4}$$

which reduces to the original Lagrangian density when all the tilde superfields vanish. Let us now define

$$\begin{aligned}\Phi_\mu(x, \theta, \bar{\theta}) &= \phi_\mu(x, \theta, \bar{\theta}) - \tilde{\phi}_\mu(x, \theta, \bar{\theta}) \\ \Lambda(x, \theta, \bar{\theta}) &= \eta(x, \theta, \bar{\theta}) - \tilde{\eta}(x, \theta, \bar{\theta}) \\ \bar{\Lambda}(x, \theta, \bar{\theta}) &= \bar{\eta}(x, \theta, \bar{\theta}) - \tilde{\bar{\eta}}(x, \theta, \bar{\theta})\end{aligned}\tag{5.5}$$

These new superfields clearly coincide with the original superfields when the tilde superfields vanish.

If we now define the connection 1-form in this superspace as

$$\Omega = \Phi_\mu dx^\mu + \Lambda d\theta + \bar{\Lambda} d\bar{\theta}\tag{5.6}$$

and require the components of the field strength along all the extra directions to vanish, then, once again we will determine

$$\begin{aligned}
\Phi_\mu(x, \theta, \bar{\theta}) &= (A_\mu - \tilde{A}_\mu) + \theta D_\mu^{(A-\tilde{A})}(c - \tilde{c}) + \bar{\theta} D_\mu^{(A-\tilde{A})}(\bar{c} - \tilde{\bar{c}}) \\
&\quad + \theta \bar{\theta} (D_\mu^{(A-\tilde{A})}(F - \tilde{F}) - [D_\mu^{(A-\tilde{A})}(c - \tilde{c}), (\bar{c} - \tilde{\bar{c}})]_+) \\
\Lambda(x, \theta, \bar{\theta}) &= (c - \tilde{c}) - \frac{1}{2} \theta [c - \tilde{c}, c - \tilde{c}]_+ + \bar{\theta} ((F - \tilde{F}) - [c - \tilde{c}, \bar{c} - \tilde{\bar{c}}]_+) \\
&\quad + \theta \bar{\theta} \left(- [F - \tilde{F}, c - \tilde{c}] + [\bar{c} - \tilde{\bar{c}}, (c - \tilde{c})^2] \right) \\
\bar{\Lambda}(x, \theta, \bar{\theta}) &= (\bar{c} - \tilde{\bar{c}}) - \theta (F - \tilde{F}) - \frac{1}{2} \bar{\theta} [\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}]_+ \\
&\quad - \theta \bar{\theta} [F - \tilde{F}, \bar{c} - \tilde{\bar{c}}]
\end{aligned} \tag{5.7}$$

The individual superfields ϕ_μ , η , $\bar{\eta}$ and $\tilde{\phi}_\mu$, $\tilde{\eta}$, $\tilde{\bar{\eta}}$, however, will contain arbitrary functions. This is the reflection of the arbitrariness in the extended BRST and anti-BRST transformations that we discussed earlier. Consistent with the discussion in sections II and III, we can choose the individual superfields as follows.

$$\begin{aligned}
\phi_\mu(x, \theta, \bar{\theta}) &= A_\mu + \theta \psi_\mu + \bar{\theta} (A_\mu^* + D_\mu^{(A-\tilde{A})}(\bar{c} - \tilde{\bar{c}})) \\
&\quad + \theta \bar{\theta} (B_\mu + D_\mu^{(A-\tilde{A})}(F - \tilde{F}) - [D_\mu^{(A-\tilde{A})}(c - \tilde{c}), (\bar{c} - \tilde{\bar{c}})]_+) \\
\tilde{\phi}_\mu(x, \theta, \bar{\theta}) &= \tilde{A}_\mu + \theta (\psi_\mu - D_\mu^{(A-\tilde{A})}(c - \tilde{c})) + \bar{\theta} A_\mu^* + \theta \bar{\theta} B_\mu \\
\eta(x, \theta, \bar{\theta}) &= c + \theta \epsilon + \bar{\theta} (c^* + (F - \tilde{F}) - [c - \tilde{c}, \bar{c} - \tilde{\bar{c}}]_+) \\
&\quad + \theta \bar{\theta} (B - [F - \tilde{F}, c - \tilde{c}] + [\bar{c} - \tilde{\bar{c}}, (c - \tilde{c})^2]) \\
\tilde{\eta}(x, \theta, \bar{\theta}) &= \tilde{c} + \theta \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) + \bar{\theta} c^* + \theta \bar{\theta} B \\
\bar{\eta}(x, \theta, \bar{\theta}) &= \bar{c} + \theta \bar{\epsilon} + \bar{\theta} \left(\bar{c}^* - \frac{1}{2} [\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}]_+ \right) \\
&\quad + \theta \bar{\theta} (\bar{B} - [F - \tilde{F}, \bar{c} - \tilde{\bar{c}}]) \\
\tilde{\bar{\eta}}(x, \theta, \bar{\theta}) &= \tilde{\bar{c}} + \theta (\bar{\epsilon} + (F - \tilde{F})) + \bar{\theta} \bar{c}^* + \theta \bar{\theta} \bar{B}
\end{aligned} \tag{5.8}$$

Here ψ_μ , ϵ , $\bar{\epsilon}$, A_μ^* , c^* , \bar{c}^* , B_μ , B and \bar{B} are arbitrary functions and can be identified with the ghosts, antighosts, and auxiliary fields introduced in sec. II in connection with the shift symmetry. In fact, it is straight forward to see that translations in the θ and $\bar{\theta}$

coordinates generate respectively the extended BRST and anti-BRST transformations of Eqs. (2.11) and (3.1) with

$$F^* = 0 = B_F \quad (5.9)$$

Furthermore, Eq. (5.8) determines the variations only in the combination $(F - \tilde{F})$ and consequently, the ghost field, ϵ_F , does not occur in the superfields. We note here that in a non gauge theory, one can introduce an additional superfield

$$\tilde{f}(x, \theta, \bar{\theta}) = \tilde{F} + \theta \epsilon_F + \bar{\theta} F^* + \theta \bar{\theta} B_F \quad (5.10)$$

to generate the additional transformations in Eqs. (2.11) and (3.1). In a gauge theory, however, the auxiliary fields occur in the antighost multiplet and, therefore, this will be artificial and would destroy the geometrical formulation of the gauge theory. We also note here that the additional transformations (or generalizations of them) can probably be generated by expanding the tilde field structures [5]. But we feel that the minimal structure in Eq. (5.8) is quite appealing and as we will show shortly the absence of the variables in Eq. (5.9) does not pose any particular difficulty in the construction of the theory.

To proceed, let us note from the structure of the superfields in Eq. (5.8) that

$$\begin{aligned} -\frac{1}{2} \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \text{Tr } \tilde{\phi}_\mu \tilde{\phi}^\mu &= \text{Tr} \left(-B_\mu \tilde{A}^\mu - A_\mu^* (\psi^\mu - D^{\mu(A-\tilde{A})} (c - \tilde{c})) \right) \\ \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \text{Tr } \tilde{\eta} \tilde{\eta} &= \text{Tr} \left(-\bar{B} \tilde{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \right. \\ &\quad \left. + B \tilde{c} - c^* (\bar{\epsilon} + (F - \tilde{F})) \right) \end{aligned} \quad (5.11)$$

Consequently, we can write

$$\begin{aligned} \tilde{\mathcal{L}}'_{gf} &= \text{Tr} \left[-B_\mu \tilde{A}^\mu - A_\mu^* (\psi^\mu - D^{\mu(A-\tilde{A})} (c - \tilde{c})) - \bar{B} \tilde{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}]_+ \right) \right. \\ &\quad \left. + B \tilde{c} - c^* (\bar{\epsilon} + (F - \tilde{F})) \right] = \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \text{Tr} \left(-\frac{1}{2} \tilde{\phi}_\mu \tilde{\phi}^\mu + \tilde{\eta} \tilde{\eta} \right) \end{aligned} \quad (5.12)$$

Being the $\theta\bar{\theta}$ component of a superfield, this gauge fixing Lagrangian is manifestly invariant under extended BRST and anti-BRST transformations. Note that the gauge fixing Lagrangian density in (5.12) differs from that in (2.12) in the F dependent terms. We will come back to this point shortly.

To write the gauge fixing Lagrangian for the original symmetry, we note that we can choose a fermionic superfield whose first component corresponds to the arbitrary fermionic gauge fixing function of Eq. (2.4). Then, we have

$$\Psi(x, \theta, \bar{\theta}) = \psi + \theta\delta\psi + \bar{\theta}\bar{\delta}\psi + \theta\bar{\theta}\delta\bar{\delta}\psi \quad (5.13)$$

In general, all four components of the superfield will be non trivial implying that if we choose as in Eq. (2.4)

$$\mathcal{L}_{gf} = \delta\psi$$

then it will not be invariant under extended anti-BRST transformations. (This follows from the fact that the $\theta\bar{\theta}$ component of the superfield in Eq. (5.13) is non vanishing in general.) However, we note that if the gauge fixing Lagrangian is both BRST and anti-BRST invariant, then the $\theta\bar{\theta}$ component of $\Psi(x, \theta, \bar{\theta})$ would vanish on-shell because when we use the equations of motion, the tilde fields vanish and the theory reduces to the original theory, where, by assumption the gauge fixing Lagrangian is both BRST and anti-BRST invariant. This can, of course, be explicitly checked for specific gauge choices. Thus, for example, for the covariant gauge fixing of Eq. (2.6), we can write

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) = & -\text{Tr } \partial_\mu \bar{\eta} \phi^\mu = \text{Tr} \left(-\partial_\mu \bar{c} A^\mu - \theta(\partial_\mu \bar{c} A^\mu - \partial_\mu \bar{c} \psi^\mu) \right. \\ & - \bar{\theta}((\partial_\mu \bar{c}^* - [\partial_\mu(\bar{c} - \tilde{c}), \bar{c} - \tilde{c}]_+) A_\mu - \partial_\mu \bar{c}(A^{\mu*} + D^{\mu(A-\tilde{A})}(\bar{c} - \tilde{c}))) \\ & - \theta\bar{\theta}((\partial_\mu \bar{B} - [\partial_\mu(F - \tilde{F}), \bar{c} - \tilde{c}] - [F - \tilde{F}, \partial_\mu(\bar{c} - \tilde{c})]) A^\mu \\ & + \partial_\mu \bar{c}(B^\mu + D^{\mu(A-\tilde{A})}(F - \tilde{F}) - [D^{\mu(A-\tilde{A})}(c - \tilde{c}), \bar{c} - \tilde{c}]_+) \\ & \left. - (\partial_\mu \bar{c}^* - [\partial_\mu(\bar{c} - \tilde{c}), \bar{c} - \tilde{c}]_+) \psi^\mu + \partial_\mu \bar{c}(A^{\mu*} + D^{\mu(A-\tilde{A})}(\bar{c} - \tilde{c})) \right) \end{aligned} \quad (5.14)$$

If we choose as the gauge fixing Lagrangian the θ component of this Lagrangian, namely (neglecting total divergences)

$$\mathcal{L}_{gf} = \partial_\mu \bar{c} \psi^\mu - \partial_\mu \bar{\epsilon} A^\mu \equiv \partial_\mu \bar{c} \psi^\mu + \partial_\mu A^\mu \bar{\epsilon} \quad (5.15)$$

it is tedious but straight forward to check that the $\theta\bar{\theta}$ component in Eq. (5.14) vanishes when we use the equations of motion. Therefore, it is clear that for an arbitrary fermionic gauge fixing function that leads to a BRST and anti-BRST gauge fixing Lagrangian, we can choose

$$\mathcal{L}_{gf} = \frac{\partial}{\partial\theta} (\delta(\bar{\theta})\Psi(x, \theta, \bar{\theta})) \quad (5.16)$$

This would, of course, be manifestly invariant under extended BRST transformations, but it will also be invariant under extended anti-BRST transformations on-shell. One can presumably add necessary auxiliary fields to make the gauge fixing Lagrangian invariant under extended anti-BRST transformations without the use of the equations of motion. However, this would take us out of the minimal geometric approach, which is the spirit of our paper.

The complete Lagrangian, which is invariant under extended BRST transformations and is also invariant under extended anti-BRST transformations on-shell can, therefore, be written as

$$\begin{aligned} \hat{\mathcal{L}} &= \mathcal{L}_o(\phi_\mu - \tilde{\phi}_\mu) + \tilde{\mathcal{L}}'_{gf} + \mathcal{L}_{gf} \\ &= \mathcal{L}_o(A_\mu - \tilde{A}_\mu) + \frac{\partial}{\partial\bar{\theta}} \frac{\partial}{\partial\theta} \text{Tr} \left(-\frac{1}{2} \tilde{\phi}_\mu \tilde{\phi}^\mu + \tilde{\eta} \tilde{\eta} \right) + \frac{\partial}{\partial\theta} (\delta(\bar{\theta})\Psi(x, \theta, \bar{\theta})) \\ &= \text{Tr} \left(-\frac{1}{4} F_{\mu\nu}(A - \tilde{A}) F^{\mu\nu}(A - \tilde{A}) - B_\mu \tilde{A}^\mu - \bar{B} \tilde{c} + B \tilde{\bar{c}} \right. \\ &\quad - \left(A_\mu^* + \frac{\delta\psi}{\delta A^\mu} \right) \psi^\mu + \left(\bar{c}^* + \frac{\delta\psi}{\delta c} \right) \epsilon - \left(c^* - \frac{\delta\psi}{\delta \bar{c}} \right) \bar{\epsilon} \\ &\quad + A_\mu^* D^{\mu(A-\tilde{A})}(c - \tilde{c}) \\ &\quad \left. + \frac{1}{2} \bar{c}^* [c - \tilde{c}, c - \tilde{c}]_+ - c^* (F - \tilde{F}) \right) \end{aligned} \quad (5.17)$$

We note that integrating out the auxiliary fields B_μ , \bar{B} and B will set the tilde fields \tilde{A}^μ , \tilde{c} and $\tilde{\bar{c}}$ to zero. Furthermore, integrating out the ghost fields for the shift symmetry,

ψ^μ , ϵ and $\bar{\epsilon}$ will determine the antifields A_μ^* , c^* and \bar{c}^* which, when substituted into the Lagrangian density will yield the BV action except for the \tilde{F} term. But we note that F and \tilde{F} are auxiliary fields and, therefore, one can trivially redefine

$$F - \tilde{F} \rightarrow F \quad (5.18)$$

The orthogonal combination $(F + \tilde{F})$ can be integrated out of the functional integral leading to an infinite constant which can be absorbed into the normalization of the path integral. With these redefinition, then, the BV action is obtained.

VI. Conclusion

The BV formalism provides a powerful quantization method within the Lagrangian formulation. Here one extends the configuration space by introducing anti-fields corresponding to all the original fields present in the theory. (These are ultimately identified with the functional derivative of a gauge fixing fermion with respect to the corresponding fields.) In this space, one defines an antibracket which generates the BRST transformations with the classical BV action as the generator. The complete quantum action is obtained as a proper solution of the master equation in a power series in \hbar which coincides with the classical BV action (up to renormalization) when there are no anomalies present. In this paper we have considered such a theory^[10] and have shown that if the gauge fixing fermion leads to a BRST and anti-BRST invariant Lagrangian, then the BV action formulated with an extended BRST symmetry is also invariant under an extended anti-BRST symmetry. We have tried to give a geometrical meaning to these extended transformations as corresponding to translations in a superspace. For a general gauge fixing fermion, we have shown that the BV action can be written in a manifestly extended BRST invariant manner in a superspace with one Grassmann coordinate. On the other hand, if the gauge fixing fermion leads to a BRST and anti-BRST invariant Lagrangian, then we have shown that the BV action can be written in a manifestly extended BRST and anti-BRST invariant manner (at least on shell) in a superspace with a pair of Grassmann coordinates. This formalism will readily generalize to other non anomalous theories. This geometric

formulation is manifestly BRST invariant and it will be interesting to see how anomalous gauge theories would fit into this formulation. This is presently under study.

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